§ Homotopy theory of Kan complexes

<u>Ex.:</u> We define the simplicial circle by the pushout

$$\begin{tikzcd} \Delta^{\{0\}} \amalg \Delta^{\{1\}} \arrow[d, phantom, "\cong" rotate=90] & \\ \partial\Delta^1 \arrow[r] \arrow[d] & \Delta^1 \arrow[d] \\ \Delta^0 \arrow[r] & \Delta^1/\partial\Delta^1 \end{tikzcd}$$

There are how many morphisms $\Delta^1/\partial\Delta^1 \to \Delta^1/\partial\Delta^1$? Only two!

For $X \in sSet$:

$$\mathrm{Hom}_{sSet}(\Delta^1/\partial\Delta^1, X) = \mathrm{Hom}_{sSet}(\Delta^1 \to X \mid \Delta^{\{0\}} \subseteq \Delta^1 \to X \text{ agrees with } \Delta^{\{1\}} \subseteq \Delta^1 \to X)$$

$$= \{x \in X_1 \mid d_1^* x = d_0^* x\}$$

For $X = \Delta^1/\partial\Delta^1$: $X_0 = \{*\}$ so that

$$\mathrm{Hom}_{sSet}(\Delta^1/\partial\Delta^1, \Delta^1/\partial\Delta^1) \cong (\Delta^1/\partial\Delta^1)_1 = \left\{[\mathrm{id}: [1] \to [1]],\ \left[\begin{array}{c}[1] \to [1] \\ 0 \mapsto 1 \\ 1 \mapsto 1\end{array}\right]\right\}$$

In topology: $\pi_1(S^1, *) \cong \mathbb{Z}$

$\Delta^1/\partial\Delta^1$ is not a Kan complex

Ex.:
$$\begin{tikzcd} & * \arrow[dr, "{[\mathrm{id}_{[1]}]}"] & \\ * \arrow[ur, "{[\mathrm{id}_{[1]}]}"] & & * \end{tikzcd}$$
can not be filled

Path components and homotopy

Def.: X Kan complex, A simplicial set

a) For $x, y \in X$: $x \sim y$ if $\exists$ 1-simplex $\Delta^1 \to X$ from x to y

(This is equivalence relation.)

$\pi_0(X) = X_0/\sim$ set of path components

b) $f, g : A \to X$ in sSet are homotopic if

$$[f] = [g] \text{ in } \pi_0(\underbrace{F(A,X)}_{\text{is Kan complex}}).$$

A homotopy $\eta : f \Rightarrow g$ is a 1-simplex $\Delta^1 \to F(A,X)$ from f to g.

Ex.: $\mathcal{C}$ small category with a terminal object $*$, $X = N(\mathcal{C})$

Then $id_X \simeq const_*$ via the nerve of

$$\begin{aligned} \mathcal{C} &\longrightarrow \mathrm{Fun}([1], \mathcal{C}) \\ c &\longmapsto (c \to *) \\ (f : c \to c') &\longmapsto \left(\begin{tikzcd} c \arrow[r] \arrow[d] & * \arrow[d] \\ c' \arrow[r] & * \end{tikzcd} \right). \end{aligned}$$

Function complexes for pairs

$A \subseteq B$ inclusion of simplicial sets, $X \subseteq Y$ inclusion of Kan complexes

$F((A,B),(X,Y))$ is the pullback

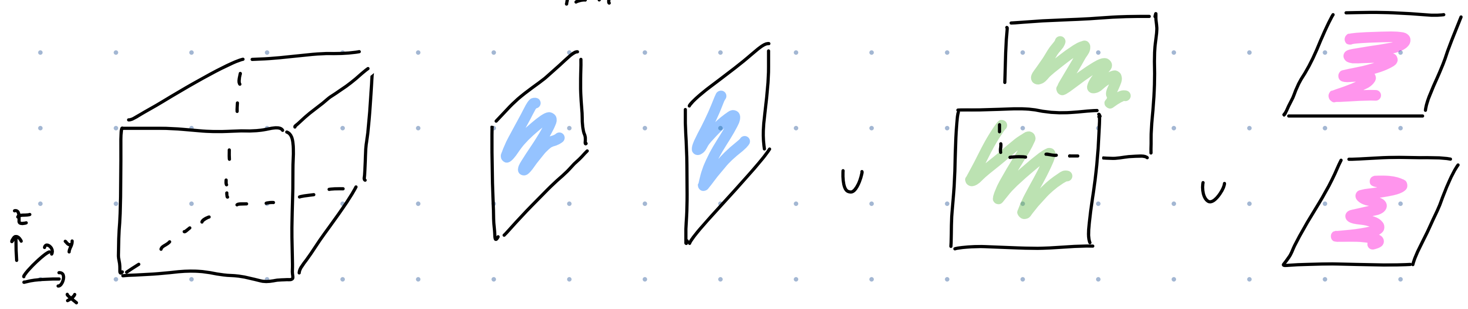
$$\begin{tikzcd} F((A,B),(X,Y)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(B,Y) \arrow[d] \\ F(A,X) \arrow[r] & F(A,Y) \end{tikzcd}$$

is Kan complex

Homotopy groups

X Kan complex, $x \in X$ point, $n \geq 1$

$\square^n = (\Delta^1)^n$, $\partial \square^n := \bigcup_{i=1}^{n} \square^{i-1} \times \partial\Delta^1 \times \square^{n-i}$



$n=3$

n^{th} homotopy group with basepoint x:

$$\pi_n(X,x) := \pi_0 F((\square^n, \partial\square^n), (X,x))$$

Fact: $\pi_n(X,x)$ is a group (abelian for $n \geq 2$): $[\alpha], [\beta] \in \pi_n(X,x)$ $\quad \alpha, \beta : \square^n \to X$

Set $(\alpha,\beta) : \Lambda^2_1 \times \square^{n-1} \to X$ by α on $\Delta^{\{0,1\}} \times \square^{n-1}$ and β on $\Delta^{\{1,2\}} \times \square^{n-1}$.

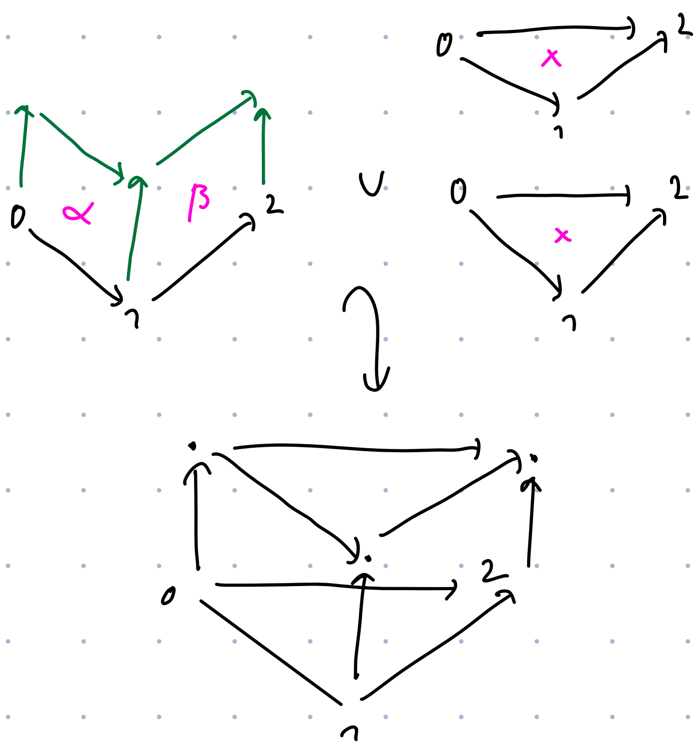
$$\begin{tikzcd}
\Lambda^2_1 \times \square^{n-1} \coprod_{\Lambda^2_1 \times \partial\square^{n-1}} \Delta^2 \times \partial\square^{n-1} \arrow[r, "{(\alpha,\beta) \cup \mathrm{const}_x}"] \arrow[d, hook] & X \\
\Delta^2 \times \square^{n-1} \arrow[ur, dashed, "\exists \theta"'] &
\end{tikzcd}$$

(since X Kan)

Composition:

$$\boxed{[\alpha] \circ [\beta] := [\theta|_{\Delta^{\{0,2\}} \times \square^{n-1}}]}$$

$n=2$



Theorem: (Whitehead Thm. for Kan complexes)

A morphism $f: X \to Y$ of Kan complexes is a homotopy equivalence iff it induces a bijection $\pi_0 X \to \pi_0 Y$ and isom. $\pi_n(X, x) \to \pi_n(Y, f(x))$ $\forall x \in X$, $n \geq 1$.

homotopy equiv. $f: X \to Y$, i.e., $\exists g: Y \to X$ s.t. $fg \simeq id_Y$, $gf \simeq id_X$

Proof uses long exact sequence for homotopy groups below.

Def.: A map $f: X \to Y$ in sSet is a Kan fibration if every commutative square

$n \geq 1$

$0 \leq i \leq n$

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

has a filler $\Delta^n \to X$.

$0 \leq i < n \rightsquigarrow$ left fibration

$0 < i \leq n \rightsquigarrow$ right fibration

$0 < i < n \rightsquigarrow$ inner fibration

Ex.:

1) $X \in sSet$ is a $\begin{cases} \text{Kan complex} \\ \text{quasi-category} \end{cases}$ iff $X \to \Delta^0$ is $\begin{cases} \text{a Kan fibration.} \\ \text{an inner fibration} \end{cases}$

2) $X \in Kan \Rightarrow X \times Y \to \Delta^0 \times Y \cong Y$ is Kan fibration $\forall Y \in sSet$

3) G group acting freely (each X_n is free G-set) on $X \in sSet$

$\Rightarrow X \to X/_G$ is Kan fibration

$$\begin{tikzcd} F_y \cong G \times \Delta^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ \Delta^n \arrow[r, "y"] & X/_G \end{tikzcd}$$

Lemma: (long exact sequence for homotopy groups)

Let $f: X \to Y$ be a Kan fibration between Kan complexes and $x \in X$.

Set $y = f(x)$ and $F = \{y\} \times_Y X$. Then $\exists$ long exact sequence of groups / pointed sets

$$\dots \to \pi_n(F,x) \to \pi_n(X,x) \to \pi_n(Y,y) \xrightarrow{\partial} \pi_{n-1}(F,x) \to \dots$$

$$\to \pi_1(Y,y) \to \pi_0(F,x) \to \pi_0(X,x) \to \pi_0(Y,y)$$

⌜$(A,a) \xrightarrow{i} (B,b) \xrightarrow{p} (C,c)$ of pointed sets is exact if $\mathrm{Im}\, i = \{z \in B \mid p(z) = c\}$⌟

Ex.: G group

EG = nerve of transport category $Tr(G)$ (action groupoid of G-set G)

objects: elements of G

morphisms: $\exists!\ g_0 \to g_1$, denoted $g_1 g_0^{-1}$

Right G-action on $Tr(G)$

$g \in G$

$$g: Tr(G) \to Tr(G)$$
$$g_0 \mapsto g_0 g$$
$$(g_0 \to g_1) \mapsto (g_0 g \xrightarrow{g_1 g_0^{-1}} g_1 g)$$

induces free action on EG:

$$(EG)_n \ni (g_0 \to g_1 \to \dots \to g_n) \mapsto (g_0 g \to \dots \to g_n g) \in (EG)_n$$

Let $BG = EG/_G$.

We show that BG is a Kan complex, in fact,

$BG = N(G)$, where G is considered as a category with one object $*$.

The functor $\overset{\cong\, h(EG)}{\mathrm{Tr}\,G} \longrightarrow G$ induces by adjunction the map

$g \longmapsto *$ $\qquad$ $EG \longrightarrow NG$

$(g_0 \to g_1) \longmapsto g_1 g_0^{-1}$ $\qquad$ $(g_0 \to \dots \to g_n) \longmapsto \left(* \xrightarrow{g_1 g_0^{-1}} * \to \dots \xrightarrow{g_n g_{n-1}^{-1}} *\right)$,

which descends to an isom. $BG = EG/_G \to NG$.

(Inverse sends $* \xrightarrow{g_1} * \to \dots \xrightarrow{g_n} *$ to $e \to g_1 \to g_2 g_1 \to \dots \to g_n \cdots g_1$)

EG is homotopy equivalent to Δ^0 since $\mathrm{Tr}\,G$ has a terminal object.

$$\rightsquigarrow \quad \underset{\substack{\shortparallel \\ 0}}{\pi_1(EG,e)} \to \pi_1(BG,*) \longrightarrow \overset{\substack{G \\ \shortparallel}}{\pi_0(G)} \longrightarrow \overset{\substack{\{[e]\} \\ \shortparallel}}{\pi_0(EG)}$$

set G as simplicial set

Have $\pi_1(BG,*) \cong G$.

Recall: $|\cdot| : \mathrm{sSet} \rightleftarrows \mathrm{Top} : \mathrm{Sing}$

Thm.: For any Kan complex X:

$$\pi_0(X) \cong \pi_0(|X|), \quad \pi_n(X,x) \cong \pi_n(|X|,x) \quad n \geq 1$$

For any topological space Y:

$$\pi_0(Y) \cong \pi_0(\mathrm{Sing}\,Y), \quad \pi_n(Y,y) \cong \pi_n(\mathrm{Sing}\,Y, y) \quad n \geq 1$$

The unit $X \longrightarrow \mathrm{Sing}(|X|)$ is a homotopy equivalence $\forall X \in \mathrm{Kan}$.

The counit $|\mathrm{Sing}\,Y| \longrightarrow Y$ is a weak homotopy equivalence $\forall Y \in \mathrm{Top}$.

§ Joyal's lifting theorem

Thm.: [Joyal's lifting theorem]

Let $p: \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasi-categories.

Then $\forall n \geq 2$:

Any of the lifting problems

$$\begin{tikzcd} \Lambda^n_0 \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d, "p"] \\ \Delta^n \arrow[r] & \mathcal{D} \end{tikzcd}$$

s.t. the 1-simplex $\Delta^{\{0,1\}} \subseteq \Lambda^n_0 \to \mathcal{C}$ is an equivalence

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d, "p"] \\ \Delta^n \arrow[r] & \mathcal{D} \end{tikzcd}$$

s.t. the 1-simplex $\Delta^{\{n-1,n\}} \subseteq \Lambda^n_n \to \mathcal{C}$ is an equivalence

has a solution.

Consequences:

Already stated last time:

("$\Rightarrow$" Extend $\underset{\overset{\shortparallel}{\Delta^{\{0\}}}}{\Lambda^n_0} \xrightarrow{x} \mathcal{C}$, $\underset{\overset{\shortparallel}{\Delta^{\{1\}}}}{\Lambda^n_1} \xrightarrow{x} \mathcal{C}$ by $id_x : \Delta^1 \to \mathcal{C}$, $n \geq 2$, use Thm.)

$\mathcal{C} \in QCat$ is ∞-groupoid iff $\mathcal{C}$ is Kan complex

New and more involved are the results involving mapping spaces.

Def.: $\mathcal{C} \in QCat$, $x, y \in \mathcal{C}$

$\mathrm{Hom}_{\mathcal{C}}(x,y)$ is the pullback of

$$\begin{tikzcd} & F(\Delta^1, \mathcal{C}) \arrow[d, "(s,t)"] \\ \Delta^0 \times \Delta^0 \arrow[r, "(x,y)"'] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

$$\mathcal{C} \times \mathcal{C} = F(\Delta^{\{0\}}, \mathcal{C}) \times F(\Delta^{\{1\}}, \mathcal{C})$$

.

Thm.: $\mathrm{Hom}_{\mathcal{C}}(x,y)$ is an ∞-groupoid

Thm.: $F: \mathcal{C} \to \mathcal{D}$ functors between quasi-categories

A nat. transformation $\eta: F \Rightarrow G$ (so $\eta: \Delta^1 \to F(\mathcal{C},\mathcal{D})$ from F to G)

is an equivalence iff $\eta_x : F(x) \to G(x)$ is an equivalence $\forall x \in \mathcal{C}$

Thm.: Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories.

TFAE:

(i) F is an equivalence in Cat_∞

(ii) $\exists G: \mathcal{D} \to \mathcal{C}$ and equivalences $\mathrm{id}_{\mathcal{C}} \Rightarrow GF$, $FG \Rightarrow \mathrm{id}_{\mathcal{D}}$

(iii) F is fully faithful, i.e., F induces homotopy equivalences of Kan complexes $\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$

$\forall x,y \in \mathcal{C}$; and

F is essentially surjective, i.e., F induces surjection

$$\pi_0(\mathrm{core}(\mathcal{C})) \longrightarrow \pi_0(\mathrm{core}(\mathcal{D})).$$

(i) $\Leftrightarrow$ (ii) last time

Rem.: $\pi_0(\mathrm{core}(\mathcal{C})) = \{x \in \mathcal{C}\}/_{x \sim y}$ if $\exists\, x \to y$ invertible